

ON THE WEAK COUPLING LIMIT OF QUANTUM MANY-BODY DYNAMICS AND THE QUANTUM BOLTZMANN EQUATION

XUWEN CHEN AND YAN GUO

ABSTRACT. The rigorous derivation of the Uehling-Uhlenbeck equation from more fundamental quantum many-particle systems is a challenging open problem in mathematics. In this paper, we exam the weak coupling limit of quantum N -particle dynamics. We assume the integral of the microscopic interaction is zero and we assume $W^{4,1}$ per-particle regularity on the coressponding BBGKY sequence so that we can rigorously commute limits and integrals. We prove that, if the BBGKY sequence does converge in some weak sense, then this weak-coupling limit must satisfy the infinite quantum Maxwell-Boltzmann hierarchy instead of the expected infinite Uehling-Uhlenbeck hierarchy, regardless of the statistics the particles obey. Our result indicates that, in order to derive the Uehling-Uhlenbeck equation, one must work with per-particle regularity bound below $W^{4,1}$.

1. INTRODUCTION

The rigorous derivation of the celebrated Uehling-Uhlenbeck equation from more fundamental quantum many-particle systems is a challenging open problem in mathematics. This problem has received a lot of attentions in recent years. In particular, Erdős, Salmhofer and Yau have given, in [4], a formal derivation of the spatially homogeneous Uehling-Uhlenbeck equation as the thermodynamic limit from the Fock space model. Around the same time, in [1, 2, 3], Benedetto, Castella, Esposito, and Pulvirenti initiated a different study of the problem with the "classical N -particle Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy" approach. Here, "classical BBGKY hierarchy" means the usual BBGKY hierarchy in \mathbb{R}^{3N+1} . Moreover, Benedetto, Castella, Esposito, and Pulvirenti consider the $N \rightarrow \infty$ limit of N particles in \mathbb{R}^3 instead of the thermodynamic limit. In this paper, we follow the classical BBGKY hierarchy approach in [1, 2, 3]. Let $t \in \mathbb{R}$, $\mathbf{x}_k = (x_1, \dots, x_k)$, $\mathbf{v}_k = (v_1, \dots, v_k) \in \mathbb{R}^{3k}$, $\varepsilon = N^{-\frac{1}{3}}$, and ϕ be an even pair interaction. We consider the following quantum BBGKY hierarchy

$$(\partial_t + \mathbf{v}_k \cdot \nabla_{\mathbf{x}_k}) f_N^{(k)} = \frac{1}{\sqrt{\varepsilon}} A_\varepsilon^{(k)} f_N^{(k)} + \frac{N}{\sqrt{\varepsilon}} B_\varepsilon^{(k+1)} f_N^{(k+1)}, \quad (1.1)$$

where

$$\begin{aligned} \frac{1}{\sqrt{\varepsilon}} A_\varepsilon^{(k)} &= \sum_{1 \leq i < j \leq k} \frac{1}{\sqrt{\varepsilon}} A_{i,j}^\varepsilon, \\ \frac{N}{\sqrt{\varepsilon}} B_\varepsilon^{(k+1)} &= \sum_{j=1}^k \frac{N}{\sqrt{\varepsilon}} B_{j,k+1}^\varepsilon, \end{aligned}$$

with

$$\begin{aligned} \frac{1}{\sqrt{\varepsilon}} A_{i,j}^\varepsilon f_N^{(k)} &= \frac{-i}{\sqrt{\varepsilon}} \frac{1}{(2\pi)^3} \sum_{\sigma=\pm 1} \sigma \int_{\mathbb{R}^3} e^{\frac{ih \cdot (x_i - x_j)}{\varepsilon}} \hat{\phi}(h) \\ &\quad \times f_N^{(k)} \left(t, \mathbf{x}_k, v_1, \dots, v_{i-1}, v_i - \sigma \frac{h}{2}, v_{i+1}, \dots, v_{j-1}, v_j + \sigma \frac{h}{2}, v_{j+1}, \dots, v_k \right) dh, \\ \frac{N}{\sqrt{\varepsilon}} B_{j,k+1}^\varepsilon f_N^{(k+1)} &= -i \frac{N}{\sqrt{\varepsilon}} \frac{1}{(2\pi)^3} \sum_{\sigma=\pm 1} \sigma \int_{\mathbb{R}^3} dx_{k+1} \int_{\mathbb{R}^3} dv_{k+1} \int_{\mathbb{R}^3} e^{\frac{ih \cdot (x_j - x_{k+1})}{\varepsilon}} \hat{\phi}(h) \\ &\quad \times f_N^{(k+1)} \left(t, \mathbf{x}_k, x_{k+1}, v_1, \dots, v_{j-1}, v_j - \sigma \frac{h}{2}, v_{j+1}, \dots, v_{k+1} + \sigma \frac{h}{2} \right) dh. \end{aligned}$$

We would like to immediately remark that, we have not assumed anything about the statistics the particles obey, though it seems that writing down hierarchy (1.1) like [2, (2.18)] suggests that we are assuming either the Bose-Einstein or the Fermi-Dirac statistics. One can check from [2, (2.18-2.22)] and [3, (2.10-2.16)] that the BBGKY hierarchy for the Bose-Einstein / Fermi-Dirac statistics and for the

Key words and phrases. Quantum Many-body Dynamics, Quantum Boltzmann Equation, BBGKY Hierarchy, Uehling-Uhlenbeck Equation, Weak Coupling.

Maxwell-Boltzmann statistics are identical, though their initial data are totally different. To be precise, the coefficient of $B_\varepsilon^{(k+1)} f_N^{(k+1)}$ is $\frac{\varepsilon^{-3}}{\sqrt{\varepsilon}}$ in [2, (2.18)] while the coefficient of $B_\varepsilon^{(k+1)} f_N^{(k+1)}$ in [3, (2.14)] is $\frac{N-k}{\sqrt{\varepsilon}}$. Since $N = \varepsilon^{-3}$, the difference $\frac{k}{\sqrt{\varepsilon}} B_{j,k+1}^\varepsilon$ must tend to zero as long as $\frac{N}{\sqrt{\varepsilon}} B_{j,k+1}^\varepsilon = \frac{\varepsilon^{-3}}{\sqrt{\varepsilon}} B_{j,k+1}^\varepsilon$ tends to a definite limit as $\varepsilon \rightarrow 0$ for every fixed k . Hence, we have not assumed anything about the statistics the particles obey.

We shall not go into the details about the rise of hierarchy (1.1). We refer the interested readers to [2] and [3]. The $\varepsilon \rightarrow 0$ limit of hierarchy (1.1) is called the weak-coupling limit of quantum many-body dynamics. As mentioned in [3], this is characterized by the fact that the potential interaction is weak in the sense that it is of order $\sqrt{\varepsilon}$ and the density of particles is 1. Therefore the number of collisions per unit time is ε^{-1} . Since the quantum mechanical cross-section in the Born approximation (justified because the potential is small) is quadratic in the potential interaction, the accumulated effect is of the order

$$\text{number of collisions} \times [\text{potential interaction}]^2 = 1/\varepsilon \times \varepsilon = 1.$$

We are concerned with the central question of identifying the weak coupling limit $\varepsilon \rightarrow 0$ (or equivalently $N \rightarrow \infty$) for such a quantum BBGKY hierarchy (1.1), even at a formal level. The expected $\varepsilon \rightarrow 0$ limit of hierarchy (1.1) is the infinite Uehling-Uhlenbeck hierarchy which is defined by

$$(\partial_t + \mathbf{v}_k \cdot \nabla_{\mathbf{x}_k}) f^{(k)} = \sum_{j=1}^k Q_{1,j,k+1} f^{(k+1)} + \sum_{j=1}^k Q_{2,j,k+2} f^{(k+2)} \quad (1.2)$$

where the two particle term $Q_{1,j,k+1}$ is given by

$$\begin{aligned} Q_{1,j,k+1} f^{(k+1)}(\mathbf{x}_k, \mathbf{v}_k) &= \int dv'_j dv_{k+1} dv'_{k+1} W(v_j, v_{k+1} | v'_j, v'_{k+1}) \\ &\quad \times \{f^{(k+1)}(\mathbf{x}_k, x_j, v_1, \dots, v'_j, \dots, v'_{k+1}) \\ &\quad - f^{(k+1)}(\mathbf{x}_k, x_j, v_1, \dots, v_{k+1})\}, \end{aligned}$$

and the three particle term $Q_{2,j,k+2}$ is given by

$$\begin{aligned} Q_{2,j,k+2} f^{(k+2)}(\mathbf{x}_k, \mathbf{v}_k) &= 8\pi^3 \theta \int dv'_j dv_{k+1} dv'_{k+1} W(v_j, v_{k+1} | v'_j, v'_{k+1}) \\ &\quad \times \{f^{(k+2)}(\mathbf{x}_k, x_j, x_j, v_1, \dots, v'_j, \dots, v'_{k+1}, v_j) \\ &\quad + f^{(k+2)}(\mathbf{x}_k, x_j, x_j, v_1, \dots, v'_j, \dots, v'_{k+1}, v_{k+1}) \\ &\quad - f^{(k+2)}(\mathbf{x}_k, x_j, x_j, v_1, \dots, v_{k+1}, v'_j) \\ &\quad - f^{(k+2)}(\mathbf{x}_k, x_j, x_j, v_1, \dots, v_{k+1}, v'_{k+1})\}. \end{aligned}$$

In the above,

$$\begin{aligned} W(v, v' | v_*, v'_*) &= \frac{1}{8\pi^2} \left[\hat{\phi}(v' - v) + \theta \hat{\phi}(v' - v_*) \right]^2 \delta(v + v_* - v' - v'_*) \\ &\quad \times \delta\left(\frac{1}{2} (v^2 + v_*^2 - (v')^2 - (v'_*)^2)\right). \end{aligned}$$

and $\theta = \pm 1$ for bosons and fermions respectively. The expected mean-field equation for the infinite Uehling-Uhlenbeck hierarchy (1.2) is exactly the Uehling-Uhlenbeck equation [8]. It arises as the special solution

$$f^{(k)}(t, \mathbf{x}_k, \mathbf{v}_k) = \prod_{j=1}^k f(t, x_j, v_j)$$

to hierarchy (1.2), provided that f satisfies

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f \\ &= \int dv_* dv'_* dv' W(v, v_* | v', v'_*) \\ &\quad \{f' f'_* (1 + 8\pi^3 \theta f) (1 + 8\pi^3 \theta f_*) - f f_* (1 + 8\pi^3 \theta f') (1 + 8\pi^3 \theta f'_*)\}. \end{aligned} \quad (1.3)$$

However, to our surprise, we find that, as long as $\{f_N^{(k+1)}\}$ is of $W^{4,1}$ per-particle regularity, the infinite Uehling-Uhlenbeck hierarchy (1.2) is not the $\varepsilon \rightarrow 0$ limit of the BBGKY hierarchy (1.1) regardless of the statistics the particles obey. We find that, regardless of the statistics the particles obey, the $\varepsilon \rightarrow 0$

limit of the BBGKY hierarchy (1.1) is the infinite quantum Maxwell-Boltzmann hierarchy coming from [3], defined as (1.4) in this paper. In fact, if one formally commutes integrals and $\lim_{\varepsilon \rightarrow 0}$ in appropriate places, one finds that, regardless of the statistics the particles obey, the formal $\varepsilon \rightarrow 0$ limit of the BBGKY hierarchy (1.1) must be the infinite quantum Maxwell-Boltzmann hierarchy (1.4) instead of the infinite Uehling-Uhlenbeck hierarchy (1.2).

To this end, we define the quantum Maxwell-Boltzmann hierarchy to be

$$(\partial_t + \mathbf{v}_k \cdot \nabla_{\mathbf{x}_k}) f^{(k)} = \sum_{j=1}^k C_{j,k+1} f^{(k+1)} \quad (1.4)$$

with

$$\begin{aligned} C_{j,k+1} f^{(k+1)} &= \frac{1}{8\pi^2} \int_{\mathbb{R}^3} dv_{j+1} \int_{\mathbb{S}^2} dS_\omega |\omega \cdot (v_j - v_{k+1})| \left| \hat{\phi}((\omega \cdot (v_j - v_{k+1})) \omega) \right|^2 \\ &\quad \times [f^{(k+1)}(t, \mathbf{x}_k, x_j, v_1, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v'_{k+1}) - \\ &\quad f^{(k+1)}(t, \mathbf{x}_k, x_j, v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_{k+1})]. \end{aligned} \quad (1.5)$$

The $C_{j,k+1}$ in (1.4) is certainly the Boltzmann collision operator. We use A^ε and B^ε to denote the inhomogeneous terms in (1.1) because neither of the $\varepsilon \rightarrow 0$ limit of A^ε nor B^ε alone gives the collision operator C . The collision operator $C_{j,k+1}$ in (1.4) arises as the $\varepsilon \rightarrow 0$ limit of a suitable composition of A^ε and B^ε .

Notice that, the mean-field equation of hierarchy (1.4) is not the Uehling-Uhlenbeck equation (1.3). If one assumes Maxwell-Boltzmann statistics as well as $\hat{\phi}(0) = 0$, a term by term convergence from (1.1) to (1.4) was rigorously established in [3]. The mean-field equation in this case is the quantum Boltzmann equation:

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f) \quad (1.6)$$

where the collision operator Q is given by

$$\begin{aligned} Q(f, f) &= \frac{1}{8\pi^2} \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{S}^2} dS_\omega |\omega \cdot (v - v_1)| \left| \hat{\phi}((\omega \cdot (v - v_1)) \omega) \right|^2 \\ &\quad [f(t, x, v') f(t, x, v'_1) - f(t, x, v) f(t, x, v_1)], \end{aligned}$$

and $\hat{\phi}$ is the Fourier transform of ϕ , and

$$v' = v - ([v - v_1] \cdot \omega) \omega, \quad v'_1 = v_1 + ([v - v_1] \cdot \omega) \omega,$$

because

$$f^{(k)}(t, \mathbf{x}_k, \mathbf{v}_k) = \prod_{j=1}^k f(t, x_j, v_j)$$

is a solution to hierarchy (1.4) provided that f solves (1.6).

In our main theorem, we assume $W^{4,1}$ per-particle regularity on the BBGKY sequence $\{f_N^{(k)}\}_{k=1}^N$ so that we can rigorously commute limits and integrals in suitable places. We are then able to prove that, if the BBGKY sequence $\{f_N^{(k)}\}_{k=1}^N$ does converge in some weak sense, then the limit sequence $\{f^{(k)} = \lim_{N \rightarrow \infty} f_N^{(k)}\}_{k=1}^\infty$ must satisfy the infinite quantum Maxwell-Boltzmann hierarchy (1.4) instead of the infinite Uehling-Uhlenbeck hierarchy (1.2), regardless of the statistics the particles obey. We work in the space $W_k^{4,1}$ which is $W^{4,1}(\mathbb{R}^{3k} \times \mathbb{R}^{3k})$ equipped with the weak topology. We work with the norm

$$\|f^{(k)}\|_{W_k^{4,1}} = \sum_{j=1}^k \sum_{m=0}^4 \left(\|\partial_{x_j}^m f^{(k)}\|_{L^1} + \|\partial_{v_j}^m f^{(k)}\|_{L^1} \right).$$

Our main theorem is the following.

Theorem 1 (Main Theorem). *Assume the interaction potential ϕ is an even Schwarz class function and satisfies the vanishing condition: $\hat{\phi}$ vanishes at the origin to at least 11th order. Suppose a subsequence of $\{\Gamma_N = \{f_N^{(k)}\}_{k=1}^N\}$ converges weakly to some $\Gamma = \{f^{(k)}\}_{k=1}^\infty$ in the following sense:*

(1) In $C([0, T], W_k^{4,1})$, we have,

$$f_N^{(k)} \rightarrow f^{(k)} \text{ as } N \rightarrow \infty,$$

(2) There is a $C > 0$ such that

$$\sup_{k, N, t \in [0, T]} \frac{1}{k} \|f_N^{(k)}\|_{W_k^{4,1}} \leq C.$$

Then $\Gamma = \{f^{(k)}\}_{k=1}^\infty$ satisfies the infinite quantum Boltzmann hierarchy (1.4), regardless of the form of the initial datum $\{f_N^{(k)}(0)\}_{k=1}^N$ or the statistics (Bose-Einstein / Fermi-Dirac / Maxwell-Boltzmann) it satisfies. In particular, $f^{(k)}$ does not satisfy the infinite Uehling-Uhlenbeck hierarchy (1.2).

We remark that Theorem 1 certainly does not imply that the Uehling-Uhlenbeck equation (1.3) is not derivable as a mean-field limit. Our result is merely an indication that, in order to derive the Uehling-Uhlenbeck equation, one must work with per-particle regularity bound below $W^{4,1}$. It is certainly an interesting question to lower the regularity requirement of Theorem 1. But we are not able to do so currently.

Before delving into the proof of Theorem 1, we would like to discuss the assumptions of Theorem 1. First of all, not only we are not specifying the statistics $\{f_N^{(k)}\}$ satisfy, we are not assuming any statistics or symmetric conditions on the limit $f^{(k)}$ either. Moreover, we do not need $f_N^{(k)}(t)$ or $f^{(k)}(t)$ to take a special form, e.g. tensor product form or quasi free form, to make Theorem 1 to hold. Compared with the work by King [6] and Landford [7]¹ on deriving the classical Boltzmann equation from models with hard spheres collision and singular potentials, the interparticle interaction ϕ we are considering here, is smooth, and hence the regularity assumption in Theorem 1 is not impossible. The proof of Theorem 1 suggests that the assumption $\int \phi = 0$ or $\hat{\phi}(0) = 0$ might actually be a necessary condition such that the quantum BBGKY hierarchy (1.1) has a $N \rightarrow \infty$ limit. See §2.4 for a discussion. For completeness, we include, in the appendix, a discussion about the cubic term of the Uehling-Uhlenbeck equation (1.3) when $\hat{\phi}(0) = 0$.

1.1. Acknowledgement. The first author would like to thank P. Germain, E. Lieb, B. Schlein, C. Sulem, and J. Yngvason for discussions related to this work.

2. PROOF OF THE MAIN THEOREM

For notational simplicity, it suffices to prove the main theorem for $k = 1$ and with the assumption that the whole sequence $\left\{\Gamma_N = \left\{f_N^{(k)}\right\}_{k=1}^N\right\}_N$ has only one limit point. Our goal is to prove the absence of cubic Uehling-Uhlenbeck terms in the limit. Let $S^{(k)}(t)$ be the solution operator to the equation

$$(\partial_t + \mathbf{v}_k \cdot \nabla_{\mathbf{x}_k}) f^{(k)} = 0.$$

We will prove that every limit point $\Gamma = \{f^{(k)}\}_{k=1}^\infty$ of $\left\{\Gamma_N = \left\{f_N^{(k)}\right\}_{k=1}^N\right\}_N$ in the sense of Theorem 1 satisfies

$$\begin{aligned} & \int J(x_1, v_1) f^{(1)}(t_1, x_1, v_1) dx_1 dv_1 \\ &= \int J(x_1, v_1) S^{(1)} f^{(1)}(0, x_1, v_1) dx_1 dv_1 \\ & \quad + \int J(x_1, v_1) \left(\int_0^{t_1} S^{(1)}(t_1 - t_2) C_{1,2} f^{(2)}(t_2, x_1, v_1) dt_2 \right) dx_1 dv_1, \end{aligned} \tag{2.1}$$

for all real test function $J(x_1, v_1)$.

To this end, we use the BBGKY hierarchy (1.1). Write hierarchy (1.1) in integral form, we have

$$\begin{aligned} f_N^{(k)}(t_k) &= S^{(k)}(t_k) f_N^{(k)}(0) + \int_0^{t_k} S^{(k)}(t_k - t_{k+1}) \frac{1}{\sqrt{\varepsilon}} A_\varepsilon^{(k)} f_N^{(k)}(t_{k+1}) dt_{k+1} \\ & \quad + \int_0^{t_k} S^{(k)}(t_k - t_{k+1}) \frac{N}{\sqrt{\varepsilon}} B_\varepsilon^{(k+1)} f_N^{(k+1)}(t_{k+1}) dt_{k+1}. \end{aligned} \tag{2.2}$$

Iterate hierarchy (2.2) once and get to

$$f_N^{(1)}(t_1) = I + II + III + IV + V, \tag{2.3}$$

¹See also [5].

where

$$\begin{aligned}
I &= S^{(1)}(t_1) f_N^{(1)}(0), \\
II &= \frac{1}{\sqrt{\varepsilon}} \int_0^{t_1} S^{(1)}(t_1 - t_2) A_\varepsilon^{(1)} f_N^{(1)}(t_2) dt_2, \\
III &= \frac{N}{\sqrt{\varepsilon}} \int_0^{t_1} S^{(1)}(t_1 - t_2) B_\varepsilon^{(2)} S^{(2)}(t_2) f_N^{(2)}(0) dt_2, \\
IV &= \frac{N}{\varepsilon} \int_0^{t_1} S^{(1)}(t_1 - t_2) B_\varepsilon^{(2)} \int_0^{t_2} S^{(2)}(t_2 - t_3) A_\varepsilon^{(2)} f_N^{(2)}(t_3) dt_3 dt_2, \\
V &= \frac{N^2}{\varepsilon} \int_0^{t_1} S^{(1)}(t_1 - t_2) B_\varepsilon^{(2)} \int_0^{t_2} S^{(2)}(t_2 - t_3) B_\varepsilon^{(3)} f_N^{(3)}(t_3) dt_3 dt_2.
\end{aligned}$$

On the one hand, iterating hierarchy (2.2) once gives the terms which are quadratic in ϕ and hence are the central part of the quantum Boltzmann hierarchy (1.4) and the Uehling-Uhlenbeck hierarchy (1.2). On the other hand, we remark that one will not obtain the infinite Uehling-Uhlenbeck hierarchy (1.2) corresponding to the Uehling-Uhlenbeck equation (1.3) even one iterates (2.2) more than once and then considers its limit as $\varepsilon \rightarrow 0$. The easiest way to see this is to notice that the new terms will not be quadratic in ϕ .

If one believes the mean-field limit

$$f_N^{(k)}(t, \mathbf{x}_j, \mathbf{v}_j) \sim \prod_{j=1}^k f(t, x_j, v_j)$$

where f satisfies some mean-field equation, then in the $\varepsilon \rightarrow 0$ limit, IV in (2.3) will generate a nonlinearity which is quadratic in f and ϕ in the mean-field equation, and V in (2.3) will produce a term which is cubic in f and quadratic in ϕ . With the above discussion in mind, alert reader can immediately tell that the main part of the proof of Theorem 1 is proving that the Boltzmann collision operator $C_{j,k+1}$ defined in (1.5) arises as the $\varepsilon \rightarrow 0$ limit of IV , and the $\varepsilon \rightarrow 0$ limit of V is zero and thus there is no Uehling-Uhlenbeck term in the limit.

Since $f_N^{(k)} \rightarrow f^{(k)}$ in the sense stated in the main theorem (Theorem 1), we know by definition that

$$\begin{aligned}
\lim_{N \rightarrow \infty} \int J(x_1, v_1) f_N^{(1)}(t_1, x_1, v_1) dx_1 dv_1 &= \int J(x_1, v_1) f^{(1)}(t_1, x_1, v_1) dx_1 dv_1, \\
\lim_{N \rightarrow \infty} \int J(x_1, v_1) S^{(1)}(t_1) f_N^{(1)}(0, x_1, v_1) dx_1 dv_1 &= \int J(x_1, v_1) S^{(1)}(t_1) f^{(1)}(0, x_1, v_1) dx_1 dv_1.
\end{aligned}$$

Moreover, it has been shown in [1, 3] that the terms II and III tend to zero as $\varepsilon \rightarrow 0$. We are left to prove the emergence of the quadratic collision kernel $C_{j,k}$ from IV and the possible cubic term V is in fact zero as $\varepsilon \rightarrow 0$.

2.1. Emergence of the Quadratic Collision Kernel. IV is the most important term since it contributes (1.5) in the limit. Recall IV

$$IV = \frac{N}{\varepsilon} \int_0^{t_1} S^{(1)}(t_1 - t_2) B_{1,2}^\varepsilon \int_0^{t_2} S^{(2)}(t_2 - t_3) A_{1,2}^\varepsilon f_N^{(2)}(t_3) dt_3 dt_2.$$

We write

$$C_{1,2}^\varepsilon f_N^{(2)} = \frac{N}{\varepsilon} B_{1,2}^\varepsilon \int_0^{t_2} S^{(2)}(t_2 - t_3) A_{1,2}^\varepsilon f_N^{(2)}(t_3) dt_3.$$

We would like to prove

$$\begin{aligned}
&\lim_{N \rightarrow \infty} \int J(x_1, v_1) \left(\int_0^{t_1} S^{(1)}(t_1 - t_2) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dt_2 \right) dx_1 dv_1 \\
&= \int J(x_1, v_1) \left(\int_0^{t_1} S^{(1)}(t_1 - t_2) C_{1,2} f^{(2)}(t_2, x_1, v_1) dt_2 \right) dx_1 dv_1,
\end{aligned} \tag{2.4}$$

and hence obtain the quadratic collision kernel which is the rightmost term in (2.1). Notice that $S^{(1)}(t_2 - t_1) J(x_1, v_1)$ is simply another test function for all t_1 and t_2 . Hence, to establish (2.4), it suffices to prove the following proposition.

Proposition 1. *Under the assumptions in Theorem 1, we have*

$$\lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dx_1 dv_1 = \int J(x_1, v_1) C_{1,2} f^{(2)}(t_2, x_1, v_1) dx_1 dv_1.$$

Proof. We prove the propopsed limit with a direct computation. We start by writing out $C_{1,2}^\varepsilon f_N^{(2)}$ step by step. First,

$$\begin{aligned}
& \int_0^{t_2} S^{(2)}(t_2 - t_3) A_{1,2}^\varepsilon f_N^{(2)}(t_3) dt_3 \\
&= \frac{(-i)}{(2\pi)^3} \sum_{\sigma_2=\pm 1} \sigma_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_2 \\
& \quad S^{(2)}(t_2 - t_3) e^{\frac{ih_2 \cdot (x_1 - x_2)}{\varepsilon}} \hat{\phi}(h_2) f_N^{(2)} \left(t_3, x_1, x_2, v_1 - \sigma_2 \frac{h_2}{2}, v_2 + \sigma_2 \frac{h_2}{2} \right) \\
&= \frac{(-i)}{(2\pi)^3} \sum_{\sigma_2=\pm 1} \sigma_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_2 e^{\frac{ih_2 \cdot (x_1 - (t_2 - t_3)v_1 - x_2 + (t_2 - t_3)v_2)}{\varepsilon}} \hat{\phi}(h_2) \\
& \quad f_N^{(2)} \left(t_3, x_1 - (t_2 - t_3)v_1, x_2 - (t_2 - t_3)v_2, v_1 - \sigma_2 \frac{h_2}{2}, v_2 + \sigma_2 \frac{h_2}{2} \right)
\end{aligned}$$

then

$$\begin{aligned}
& B_{1,2}^\varepsilon \int_0^{t_2} S^{(2)}(t_2 - t_3) A_{1,2}^\varepsilon f^{(2)}(t_3) dt_3 \\
&= \frac{(-i)^2}{(2\pi)^6} \sum_{\sigma_1, \sigma_2=\pm 1} \sigma_1 \sigma_2 \int_{\mathbb{R}^3} dx_2 \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{R}^3} dh_1 e^{\frac{ih_1 \cdot (x_1 - x_2)}{\varepsilon}} \hat{\phi}(h_1) \\
& \quad \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_2 e^{\frac{ih_2 \cdot [x_1 - (t_2 - t_3)(v_1 - \sigma_1 \frac{h_1}{2}) - x_2 + (t_2 - t_3)(v_2 + \sigma_1 \frac{h_1}{2})]}{\varepsilon}} \hat{\phi}(h_2) \\
& \quad f_N^{(2)}(t_3, x_1 - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), x_2 - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right), \\
& \quad v_1 - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{h_1}{2}).
\end{aligned}$$

Rearrange, we have

$$\begin{aligned}
&= \frac{(-i)^2}{(2\pi)^6} \sum_{\sigma_1, \sigma_2=\pm 1} \sigma_1 \sigma_2 \int_{\mathbb{R}^3} dx_2 \int_{\mathbb{R}^3} dv_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad e^{\frac{ih_1 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot [x_1 - x_2 - (t_2 - t_3)(v_1 - v_2 - \sigma_1 h_1)]}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\
& \quad f_N^{(2)}(t_3, x_1 - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), x_2 - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right), \\
& \quad v_1 - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{h_1}{2}).
\end{aligned}$$

So

$$\begin{aligned}
& \int J(x_1, v_1) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dx_1 dv_1 \\
&= \frac{N}{\varepsilon} \frac{(-i)^2}{(2\pi)^6} \sum_{\sigma_1, \sigma_2=\pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J(x_1, v_1) e^{\frac{ih_1 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot [x_1 - x_2 - (t_2 - t_3)(v_1 - v_2 - \sigma_1 h_1)]}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\
& \quad f_N^{(2)}(t_3, x_1 - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), x_2 - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right), \\
& \quad v_1 - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{h_1}{2}).
\end{aligned}$$

The h_1 and h_2 integrals are highly oscillatory. We change variables to move the h 's away from $f_N^{(2)}$: first the x part,

$$\begin{aligned}
x_{1,new} &= x_{1,old} - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), \\
x_{2,new} &= x_{2,old} - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right),
\end{aligned}$$

which gives

$$\begin{aligned}
& \int J(x_1, v_1) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dx_1 dv_1 \\
&= \frac{N(-i)^2}{\varepsilon(2\pi)^6} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J\left(x_1 + (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2}\right), v_1\right) \\
& \quad e^{\frac{ih_1 \cdot (x_1 - x_2 + (t_2 - t_3)(v_1 - v_2 - \sigma_1 h_1))}{\varepsilon}} e^{\frac{ih_2 \cdot (x_1 - x_2)}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\
& \quad f_N^{(2)}\left(t_3, x_1, x_2, v_1 - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{h_1}{2}\right).
\end{aligned}$$

Then the v part

$$\begin{aligned}
v_{1,new} &= v_{1,old} - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, \\
v_{2,new} &= v_{2,old} + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{h_1}{2},
\end{aligned}$$

which yields

$$\begin{aligned}
& \int J(x_1, v_1) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dx_1 dv_1 \\
&= \frac{N(-i)^2}{\varepsilon(2\pi)^6} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J\left(x_1 + (t_2 - t_3) \left(v_1 + \sigma_1 \frac{h_2}{2}\right), v_1 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{h_1}{2}\right) \\
& \quad e^{\frac{ih_1 \cdot (x_1 - x_2 + (t_2 - t_3)(v_1 - v_2 + \sigma_2 h_2))}{\varepsilon}} e^{\frac{ih_2 \cdot (x_1 - x_2)}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) f_N^{(2)}(t_3, x_1, x_2, v_1, v_2).
\end{aligned}$$

To evaluate the above integral, we substitute like [1, (2.15)]

$$t_3 = t_2 - \varepsilon s_1, \quad h_1 = \varepsilon \xi_1 - h_2,$$

and have

$$\begin{aligned}
& \int J(x_1, v_1) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dx_1 dv_1 \\
&= \frac{\varepsilon^4 (-i)^2}{\varepsilon^4 (2\pi)^6} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_2 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J\left(x_1 + \varepsilon s_1 \left(v_1 + \sigma_1 \frac{h_2}{2}\right), v_1 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{(\varepsilon \xi_1 - h_2)}{2}\right) \\
& \quad e^{\frac{i(\varepsilon \xi_1 - h_2) \cdot (x_1 - x_2 + \varepsilon s_1 (v_1 - v_2 + \sigma_2 h_2))}{\varepsilon}} e^{\frac{ih_2 \cdot (x_1 - x_2)}{\varepsilon}} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) f_N^{(2)}(t_2 - \varepsilon s_1, x_1, x_2, v_1, v_2)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&= \frac{(-i)^2}{(2\pi)^6} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_2 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J\left(x_1 + \varepsilon s_1 \left(v_1 + \sigma_1 \frac{h_2}{2}\right), v_1 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{(\varepsilon \xi_1 - h_2)}{2}\right) \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\
& \quad e^{i\xi_1 \cdot (x_1 - x_2)} e^{i(\varepsilon \xi_1 - h_2) \cdot s_1 (v_1 - v_2 + \sigma_2 h_2)} f_N^{(2)}(t_2 - \varepsilon s_1, x_1, x_2, v_1, v_2).
\end{aligned} \tag{2.5}$$

Taking the $\varepsilon \rightarrow 0$ limit (justified in §2.3), we arrive at

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dx_1 dv_1 \\
&= \frac{(-i)^2}{(2\pi)^6} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_2 \int_0^{+\infty} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J(x_1, v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2}) \hat{\phi}(-h_2) \hat{\phi}(h_2) \\
& \quad e^{i\xi_1 \cdot (x_1 - x_2)} e^{-ih_2 \cdot s_1 (v_1 - v_2 + \sigma_2 h_2)} f^{(2)}(t_2, x_1, x_2, v_1, v_2)
\end{aligned} \tag{2.6}$$

Using the fact that

$$\int_{\mathbb{R}^3} e^{i\xi_1 \cdot (x_1 - x_2)} d\xi_1 = (2\pi)^3 \delta(x_1 - x_2),$$

(2.6) becomes

$$\begin{aligned}
&= \frac{(-i)^2}{(2\pi)^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_2 \int_0^{+\infty} ds_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J(x_1, v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2}) \hat{\phi}(-h_2) \hat{\phi}(h_2) \delta(x_1 - x_2) \\
& \quad e^{-ih_2 \cdot s_1 (v_1 - v_2 + \sigma_2 h_2)} f^{(2)}(t_2, x_1, x_2, v_1, v_2) \\
&= \frac{(-i)^2}{(2\pi)^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int dx_1 \int d\mathbf{v}_2 \int_0^{+\infty} ds_1 \int_{\mathbb{R}^3} dh_2 \\
& \quad J(x_1, v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2}) \hat{\phi}(-h_2) \hat{\phi}(h_2) e^{-ih_2 \cdot s_1 (v_1 - v_2 + \sigma_2 h_2)} f^{(2)}(t_2, x_1, x_1, v_1, v_2)
\end{aligned}$$

Put in spherical coordinates for the dh_2 integration: we let $h_2 = r\omega$, where $r \in \mathbb{R}^+$ and $\omega \in \mathbb{S}^2$, to get

$$\begin{aligned}
& \frac{(-i)^2}{(2\pi)^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int dx_1 \int d\mathbf{v}_2 \int_0^{+\infty} ds_1 \int_0^\infty r^2 dr \int_{\mathbb{S}^2} dS_\omega \\
& \quad J(x_1, v_1 + \sigma_2 \frac{r\omega}{2} - \sigma_1 \frac{r\omega}{2}) \left| \hat{\phi}(r\omega) \right|^2 e^{-ir\omega \cdot s_1 (v_1 - v_2 + \sigma_2 r\omega)} f^{(2)}(t_2, x_1, x_1, v_1, v_2).
\end{aligned}$$

Substitute $u = rs_1$,

$$\begin{aligned}
&= \frac{(-i)^2}{(2\pi)^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int dx_1 \int d\mathbf{v}_2 \int_0^{+\infty} du \\
& \quad \int_0^\infty r dr \int_{\mathbb{S}^2} dS_\omega J(x_1, v_1 + \sigma_2 \frac{r\omega}{2} - \sigma_1 \frac{r\omega}{2}) \left| \hat{\phi}(r\omega) \right|^2 \\
& \quad e^{-iu[(v_1 - v_2) \cdot \omega + \sigma_2 r]} f^{(2)}(t_2, x_1, x_1, v_1, v_2).
\end{aligned} \tag{2.7}$$

Write $J(x_1, v_1 + \sigma_2 \frac{r\omega}{2} - \sigma_1 \frac{r\omega}{2}) = g((\sigma_2 - \sigma_1) \frac{r\omega}{2})$ for short at the moment. Notice that in the middle of (2.7), we have

$$\begin{aligned}
& \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \left(\int_0^{+\infty} du \int_{\mathbb{S}^2} dS_\omega g((\sigma_2 - \sigma_1) \frac{r\omega}{2}) \left| \hat{\phi}(r\omega) \right|^2 e^{-iu[(v_1 - v_2) \cdot \omega + \sigma_2 r]} \right) \\
&= \left(\int_0^{+\infty} du \int_{\mathbb{S}^2} dS_\omega g(0) \left| \hat{\phi}(r\omega) \right|^2 e^{-iu[(v_1 - v_2) \cdot \omega + r]} \right) \\
& \quad - \left(\int_0^{+\infty} du \int_{\mathbb{S}^2} dS_\omega g(r\omega) \left| \hat{\phi}(r\omega) \right|^2 e^{-iu[(v_1 - v_2) \cdot \omega + r]} \right) \\
& \quad + \left(\int_0^{+\infty} du \int_{\mathbb{S}^2} dS_\omega g(0) \left| \hat{\phi}(r\omega) \right|^2 e^{-iu[(v_1 - v_2) \cdot \omega - r]} \right) \\
& \quad - \left(\int_0^{+\infty} du \int_{\mathbb{S}^2} dS_\omega g(-r\omega) \left| \hat{\phi}(r\omega) \right|^2 e^{-iu[(v_1 - v_2) \cdot \omega - r]} \right) \\
&= A - B + C - D.
\end{aligned} \tag{2.8}$$

Do the substitution, $\omega_{new} = -\omega_{old}$ in terms C and D , we then find that $C = \bar{A}$ and $D = \bar{B}$. So (2.8) is actually

$$\begin{aligned} &= \int_{\mathbb{S}^2} dS_\omega \left(g(0) \left| \hat{\phi}(r\omega) \right|^2 2 \operatorname{Re} \int_0^{+\infty} du e^{-iu[(v_1-v_2)\cdot\omega+r]} \right) \\ &\quad - \int_{\mathbb{S}^2} dS_\omega \left(g(r\omega) \left| \hat{\phi}(r\omega) \right|^2 2 \operatorname{Re} \int_0^{+\infty} du e^{-iu[(v_1-v_2)\cdot\omega+r]} \right), \end{aligned}$$

where

$$\begin{aligned} \operatorname{Re} \int_0^{+\infty} du e^{-iu[(v_1-v_2)\cdot\omega+r]} &= \operatorname{Re} \int_{-\infty}^{+\infty} du e^{-iu[(v_1-v_2)\cdot\omega+r]} H(u) \\ &= \operatorname{Re} \hat{H}([(v_1-v_2)\cdot\omega+r]) = \pi \delta((v_1-v_2)\cdot\omega+r). \end{aligned}$$

if we denote the Heaviside function by H .

Putting the above computation of (2.8) into (2.7), we have

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) C_{1,2}^\varepsilon f_N^{(2)}(t_2, x_1, v_1) dx_1 dv_1 \\ &= \frac{-2}{8\pi^2} \int dx_1 \int d\mathbf{v}_2 \int_0^\infty r dr \int_{\mathbb{S}^2} dS_\omega [J(x_1, v_1) - J(x_1, v_1 + r\omega)] \\ &\quad \delta((v_1 - v_2) \cdot \omega + r) \left| \hat{\phi}(r\omega) \right|^2 f^{(2)}(t_2, x_1, x_1, v_1, v_2) \end{aligned}$$

Insert a Heaviside function $H(r)$ to do the dr integral,

$$\begin{aligned} &= \frac{2}{8\pi^2} \int dx_1 \int d\mathbf{v}_2 \int_{-\infty}^\infty H(r) r dr \int_{\mathbb{S}^2} dS_\omega \\ &\quad (J(x_1, v_1 + r\omega) - J(x_1, v_1)) \\ &\quad \pi \delta((v_1 - v_2) \cdot \omega + r) \left| \hat{\phi}(r\omega) \right|^2 f^{(2)}(t_2, x_1, x_1, v_1, v_2) \\ &= \frac{2}{8\pi^2} \int dx_1 \int d\mathbf{v}_2 \int_{\mathbb{S}^2} dS_\omega (J(x_1, v_1 - [(v_1 - v_2) \cdot \omega] \omega) - J(x_1, v_1)) \\ &\quad H(-(v_1 - v_2) \cdot \omega) (-(v_1 - v_2) \cdot \omega) \left| \hat{\phi}([(v_1 - v_2) \cdot \omega] \omega) \right|^2 f^{(2)}(t_2, x_1, x_1, v_1, v_2) \end{aligned}$$

That is

$$\begin{aligned} &= \frac{1}{8\pi^2} \int dx_1 \int d\mathbf{v}_2 \int_{\mathbb{S}^2} dS_\omega (J(x_1, v_1 - [(v_1 - v_2) \cdot \omega] \omega) - J(x_1, v_1)) \\ &\quad |(v_1 - v_2) \cdot \omega| \left| \hat{\phi}([(v_1 - v_2) \cdot \omega] \omega) \right|^2 f^{(2)}(t_2, x_1, x_1, v_1, v_2), \end{aligned}$$

which is exactly

$$\int J(x_1, v_1) C_{1,2} f^{(2)}(t_2, x_1, v_1) dx_1 dv_1.$$

Whence we conclude the proof of Proposition 1. \square

2.2. The Cubic Term is Zero. Here, we investigate the limit of

$$V = \frac{N^2}{\varepsilon} \int_0^{t_1} S^{(1)}(t_1 - t_2) B_\varepsilon^{(2)} \int_0^{t_2} S^{(2)}(t_2 - t_3) B_\varepsilon^{(3)} f_N^{(3)}(t_3) dt_3 dt_2.$$

We write

$$\begin{aligned} Q_{1,3}^\varepsilon f_N^{(3)} &= \frac{N^2}{\varepsilon} B_{1,2}^\varepsilon \int_0^{t_2} S^{(2)}(t_2 - t_3) (B_{1,3}^\varepsilon + B_{2,3}^\varepsilon) f_N^{(3)}(t_3) dt_3 \\ &= Q_{1,3}^{\varepsilon,1} f_N^{(3)} + Q_{1,3}^{\varepsilon,2} f_N^{(3)}. \end{aligned}$$

If the $\varepsilon \rightarrow 0$ limit of $Q_{1,3}^\varepsilon f_N^{(3)}$ is nonzero, it will correspond to a cubic nonlinearity in the mean-field equation. On the one hand, as we remarked earlier in the paper, for the Uehling-Uhlenbeck equation (1.3) to rise as the mean-field equation, $\lim_{\varepsilon \rightarrow 0} Q_{1,3}^\varepsilon f_N^{(3)}$ must not be zero. On the other hand, $\lim_{\varepsilon \rightarrow 0} Q_{1,3}^\varepsilon f_N^{(3)}$ has to be zero for Theorem 1 to hold. Hence, we compute $\lim_{\varepsilon \rightarrow 0} Q_{1,3}^{\varepsilon,1} f_N^{(3)}$ and $\lim_{\varepsilon \rightarrow 0} Q_{1,3}^{\varepsilon,2} f_N^{(3)}$ in complete detail.

2.2.1. *Treatment of $Q_{1,3}^{\varepsilon,1}f_N^{(3)}$.* We prove that the limit

$$\lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)}(t_2) dx_1 dv_1 = 0$$

by direct computation. Since the proposed limit is zero, we drop the prefactor $\frac{(-i)}{(2\pi)^3}$ in B_ε so that we do not need to keep track of it. We write

$$\begin{aligned} & \int_0^{t_2} S^{(2)}(t_2 - t_3) B_{1,3}^\varepsilon f_N^{(3)}(t_3) dt_3 \\ = & \sum_{\sigma_2 = \pm 1} \sigma_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dx_3 \int_{\mathbb{R}^3} dv_3 \int_{\mathbb{R}^3} dh_2 \\ & S^{(2)}(t_2 - t_3) e^{\frac{ih_2 \cdot (x_1 - x_3)}{\varepsilon}} \hat{\phi}(h_2) f_N^{(3)} \left(t_3, x_1, x_2, x_3, v_1 - \sigma_2 \frac{h_2}{2}, v_2, v_3 + \sigma_2 \frac{h_2}{2} \right). \end{aligned}$$

Different from the quadratic term treated in §2.1, $S^{(2)}$ has no effect on (x_3, v_3) , so it becomes

$$\begin{aligned} = & \sum_{\sigma_2 = \pm 1} \sigma_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dx_3 \int_{\mathbb{R}^3} dv_3 \int_{\mathbb{R}^3} dh_2 e^{\frac{ih_2 \cdot [(x_1 - (t_2 - t_3)v_1] - x_3]}{\varepsilon}} \hat{\phi}(h_2) \\ & \times f_N^{(3)} \left(t_3, x_1 - (t_2 - t_3)v_1, x_2 - (t_2 - t_3)v_2, x_3, v_1 - \sigma_2 \frac{h_2}{2}, v_2, v_3 + \sigma_2 \frac{h_2}{2} \right). \end{aligned}$$

Then

$$\begin{aligned} & \frac{N^2}{\varepsilon} B_{1,2}^\varepsilon \int_0^{t_2} S^{(2)}(t_2 - t_3) B_{1,3}^\varepsilon f_N^{(3)}(t_3) dt_3 \\ = & \frac{N^2}{\varepsilon} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dx_2 \int_{\mathbb{R}^3} dx_3 \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{R}^3} dv_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\ & e^{\frac{ih_1 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot [(x_1 - (t_2 - t_3)(v_1 - \sigma_1 \frac{h_1}{2})) - x_3]}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\ & \times f_N^{(3)}(t_3, x_1 - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), x_2 - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right), x_3, \\ & v_1 - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_1 \frac{h_1}{2}, v_3 + \sigma_2 \frac{h_2}{2}), \end{aligned}$$

thus

$$\begin{aligned} & \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 \\ = & \frac{1}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\ & e^{\frac{ih_1 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot [(x_1 - (t_2 - t_3)(v_1 - \sigma_1 \frac{h_1}{2})) - x_3]}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) J(x_1, v_1) \\ & \times f_N^{(3)}(t_3, x_1 - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), x_2 - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right), x_3, \\ & v_1 - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_1 \frac{h_1}{2}, v_3 + \sigma_2 \frac{h_2}{2}). \end{aligned}$$

We move all h 's away from $f_N^{(3)}$. First, we substitute the x -part with

$$\begin{aligned} x_{1,new} &= x_{1,old} - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right) \\ x_{2,new} &= x_{2,old} - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right) \end{aligned}$$

which gives

$$\begin{aligned}
& \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 \\
&= \frac{1}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{ih_1 \cdot (x_1 + (t_2 - t_3)(v_1 - \sigma_1 \frac{h_1}{2}) - x_2 - (t_2 - t_3)(v_2 + \sigma_1 \frac{h_1}{2}))}{\varepsilon}} e^{\frac{ih_2 \cdot (x_1 - x_3)}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\
& J(x_1 + (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), v_1) \\
& f_N^{(3)}(t_3, x_1, x_2, x_3, v_1 - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_1 \frac{h_1}{2}, v_3 + \sigma_2 \frac{h_2}{2}).
\end{aligned}$$

Then the v -substitution:

$$v_{1,new} = v_{1,old} - \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_1}{2}, \quad v_{2,new} = v_{2,old} + \sigma_1 \frac{h_1}{2}, \quad v_{3,new} = v_{3,old} + \sigma_2 \frac{h_2}{2}$$

gives

$$\begin{aligned}
& \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 \\
&= \frac{1}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{ih_1 \cdot (x_1 + (t_2 - t_3)(v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - (t_2 - t_3)v_2)}{\varepsilon}} e^{\frac{ih_2 \cdot (x_1 - x_3)}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\
& J(x_1 + (t_2 - t_3) \left(v_1 + \sigma_2 \frac{h_2}{2} \right), v_1 + \sigma_2 \frac{h_2}{2} + \sigma_1 \frac{h_1}{2}) \\
& f_N^{(3)}(t_3, x_1, x_2, x_3, v_1, v_2, v_3).
\end{aligned}$$

Redo the change of variable:

$$t_3 = t_2 - \varepsilon s_1, \quad h_1 = \varepsilon \xi_1 - h_2,$$

we then have

$$\begin{aligned}
& \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 \\
&= \frac{\varepsilon^4}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{i(\varepsilon \xi_1 - h_2) \cdot (x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - \varepsilon s_1 v_2)}{\varepsilon}} e^{\frac{ih_2 \cdot (x_1 - x_3)}{\varepsilon}} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\
& J(x_1 + \varepsilon s_1 \left(v_1 + \sigma_2 \frac{h_2}{2} \right), v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2} + \sigma_1 \frac{\varepsilon \xi_1}{2}) \\
& f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_3, v_1, v_2, v_3).
\end{aligned}$$

Write out the phase,

$$\begin{aligned}
&= \frac{1}{\varepsilon^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{i\varepsilon \xi_1 \cdot (x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - \varepsilon s_1 v_2)}{\varepsilon}} e^{\frac{-ih_2 \cdot (\varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - \varepsilon s_1 v_2)}{\varepsilon}} e^{\frac{-ih_2 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot (x_1 - x_3)}{\varepsilon}} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\
& J(x_1 + \varepsilon s_1 \left(v_1 + \sigma_2 \frac{h_2}{2} \right), v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2} + \sigma_1 \frac{\varepsilon \xi_1}{2}) \\
& f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_3, v_1, v_2, v_3).
\end{aligned}$$

Rearrange the phase,

$$\begin{aligned}
&= \frac{1}{\varepsilon^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
&\quad e^{i\xi_1 \cdot (x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - \varepsilon s_1 v_2)} e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} e^{\frac{ih_2 \cdot (x_2 - x_3)}{\varepsilon}} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\
&\quad J(x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}), v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2} + \sigma_1 \frac{\varepsilon \xi_1}{2}) \\
&\quad f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_3, v_1, v_2, v_3)
\end{aligned}$$

Now we need to perform one more change of variable to take care of $[ih_2 \cdot (x_2 - x_3)]/\varepsilon$. We do

$$x_{3,old} = x_2 - \varepsilon x_{3,new}, \quad (2.9)$$

and arrive at

$$\begin{aligned}
&\int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 \\
&= \frac{\varepsilon^3}{\varepsilon^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
&\quad e^{i\xi_1 \cdot (x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - \varepsilon s_1 v_2)} e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} e^{ih_2 \cdot x_3} \\
&\quad \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) J(x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}), v_1 + \sigma_2 \frac{h_2}{2}) \\
&\quad f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_2 - \varepsilon x_3, v_1, v_2, v_3)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
&\quad e^{i\xi_1 \cdot (x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - \varepsilon s_1 v_2)} e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} e^{ih_2 \cdot x_3} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\
&\quad J(x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}), v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2} + \sigma_1 \frac{\varepsilon \xi_1}{2}) \\
&\quad f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_2 - \varepsilon x_3, v_1, v_2, v_3).
\end{aligned} \quad (2.10)$$

Taking the $\varepsilon \rightarrow 0$ limit inside, which is justified in §2.3, we have

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 \\
&= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^\infty ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
&\quad e^{i\xi_1 \cdot (x_1 - x_2)} e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} e^{ih_2 \cdot x_3} \hat{\phi}(-h_2) \hat{\phi}(h_2) \\
&\quad J(x_1, v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2}) f^{(3)}(t_2, x_1, x_2, x_2, v_1, v_2, v_3).
\end{aligned} \quad (2.11)$$

Do the dx_3 (not $d\mathbf{x}_3$) integration,

$$\begin{aligned}
&= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_3 \int_0^\infty ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
&\quad e^{i\xi_1 \cdot (x_1 - x_2)} e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} \delta(h_2) \hat{\phi}(-h_2) \hat{\phi}(h_2) \\
&\quad J(x_1, v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2}) f^{(3)}(t_2, x_1, x_2, x_2, v_1, v_2, v_3)
\end{aligned}$$

Do the $dx_2 d\xi_1 dh_2$ integration,

$$= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int dx_1 \int d\mathbf{v}_3 \int_0^\infty ds_1 \left| \hat{\phi}(0) \right|^2 J(x_1, v_1) f^{(3)}(t_2, x_1, x_1, x_1, v_1, v_2, v_3)$$

Since $\hat{\phi}(0) = 0$, the above is zero and hence

$$\lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 = 0.$$

Notice that if $\hat{\phi}(0) \neq 0$, the ds_1 integral yields an infinity. We formally see that it is necessary for ϕ to have zero integration in order to have the quantum Boltzmann hierarchy (1.4) and hence the quantum Boltzmann equation (1.6). We will go back to (2.10) in §2.4 to discuss more about this. It is natural to wonder if $Q_{1,3}^{\varepsilon,2}f_N^{(3)}$ will carry a negative sign and hence cancel out $Q_{1,3}^{\varepsilon,1}f_N^{(3)}$. Such a guess is not true. The term

$$\lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1$$

actually equals to (2.11) with no sign difference. (See (2.13).) In below we treat $Q_{1,3}^{\varepsilon,2}f_N^{(3)}$.

2.2.2. *Treatment of $Q_{1,3}^{\varepsilon,2}f_N^{(3)}$.* We write

$$\begin{aligned} & \int_0^{t_2} S^{(2)}(t_2 - t_3) B_{2,3}^{\varepsilon} f_N^{(3)}(t_3) dt_3 \\ = & \sum_{\sigma_2 = \pm 1} \sigma_2 \int_0^{t_2} S^{(2)}(t_2 - t_3) dt_3 \int_{\mathbb{R}^3} dx_3 \int_{\mathbb{R}^3} dv_3 \int_{\mathbb{R}^3} dh_2 e^{\frac{ih_2 \cdot (x_2 - x_3)}{\varepsilon}} \hat{\phi}(h_2) \\ & f_N^{(3)} \left(t_3, x_1, x_2, x_3, v_1, v_2 - \sigma_2 \frac{h_2}{2}, v_3 + \sigma_2 \frac{h_2}{2} \right) \\ = & \sum_{\sigma_2 = \pm 1} \sigma_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dx_3 \int_{\mathbb{R}^3} dv_3 \int_{\mathbb{R}^3} dh_2 e^{\frac{ih_2 \cdot [(x_2 - (t_2 - t_3)v_2] - x_3]}{\varepsilon}} \hat{\phi}(h_2) \\ & f_N^{(3)} \left(t_3, x_1 - (t_2 - t_3)v_1, x_2 - (t_2 - t_3)v_2, x_3, v_1, v_2 - \sigma_2 \frac{h_2}{2}, v_3 + \sigma_2 \frac{h_2}{2} \right) \end{aligned}$$

then

$$\begin{aligned} & \frac{N^2}{\varepsilon} B_{1,2}^{\varepsilon} \int_0^{t_2} S^{(2)}(t_2 - t_3) B_{2,3}^{\varepsilon} f_N^{(3)}(t_3) dt_3 \\ = & \frac{N^2}{\varepsilon} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dx_2 \int_{\mathbb{R}^3} dx_3 \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{R}^3} dv_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\ & e^{\frac{ih_1 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot [(x_2 - (t_2 - t_3)(v_2 + \sigma_1 \frac{h_1}{2})) - x_3]}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\ & f_N^{(3)} \left(t_3, x_1 - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), x_2 - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right), x_3, \right. \\ & \left. v_1 - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_1 \frac{h_1}{2} - \sigma_2 \frac{h_2}{2}, v_3 + \sigma_2 \frac{h_2}{2} \right) \end{aligned}$$

thus

$$\begin{aligned} & \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 \\ = & \frac{1}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\ & e^{\frac{ih_1 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot [(x_2 - (t_2 - t_3)(v_2 + \sigma_1 \frac{h_1}{2})) - x_3]}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) J(x_1, v_1) \\ & f_N^{(3)} \left(t_3, x_1 - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), x_2 - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right), x_3, \right. \\ & \left. v_1 - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_1 \frac{h_1}{2} - \sigma_2 \frac{h_2}{2}, v_3 + \sigma_2 \frac{h_2}{2} \right). \end{aligned}$$

Again, we change variables to move all h 's away from $f_N^{(3)}$. We use the new x -variables:

$$\begin{aligned} x_{1,new} &= x_{1,old} - (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right) \\ x_{2,new} &= x_{2,old} - (t_2 - t_3) \left(v_2 + \sigma_1 \frac{h_1}{2} \right) \end{aligned}$$

which gives

$$\begin{aligned}
& \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 \\
&= \frac{1}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{ih_1 \cdot (x_1 + (t_2 - t_3)(v_1 - \sigma_1 \frac{h_1}{2}) - x_2 - (t_2 - t_3)(v_2 + \sigma_1 \frac{h_1}{2}))}{\varepsilon}} e^{\frac{ih_2 \cdot (x_2 - x_3)}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\
& J(x_1 + (t_2 - t_3) \left(v_1 - \sigma_1 \frac{h_1}{2} \right), v_1) \\
& f_N^{(3)}(t_3, x_1, x_2, x_3, v_1 - \sigma_1 \frac{h_1}{2}, v_2 + \sigma_1 \frac{h_1}{2} - \sigma_2 \frac{h_2}{2}, v_3 + \sigma_2 \frac{h_2}{2}).
\end{aligned}$$

Then the new velocity variables:

$$v_{1,new} = v_{1,old} - \sigma_1 \frac{h_1}{2}, \quad v_{2,new} = v_{2,old} + \sigma_1 \frac{h_1}{2} - \sigma_2 \frac{h_2}{2}, \quad v_{3,new} = v_{3,old} + \sigma_2 \frac{h_2}{2}$$

gives

$$\begin{aligned}
& \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 \\
&= \frac{1}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{t_2} dt_3 \int_{\mathbb{R}^3} dh_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{ih_1 \cdot (x_1 + (t_2 - t_3)v_1 - x_2 - (t_2 - t_3)(v_2 + \sigma_2 \frac{h_2}{2}))}{\varepsilon}} e^{\frac{ih_2 \cdot (x_2 - x_3)}{\varepsilon}} \hat{\phi}(h_1) \hat{\phi}(h_2) \\
& J(x_1 + (t_2 - t_3)v_1, v_1 + \sigma_1 \frac{h_1}{2}) f_N^{(3)}(t_3, x_1, x_2, x_3, v_1, v_2, v_3).
\end{aligned}$$

With the change of variables

$$t_3 = t_2 - \varepsilon s_1, \quad h_1 = \varepsilon \xi_1 - h_2,$$

we then have

$$\begin{aligned}
& \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 \\
&= \frac{\varepsilon^4}{\varepsilon^7} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{i(\varepsilon \xi_1 - h_2) \cdot (x_1 + \varepsilon s_1 v_1 - x_2 - \varepsilon s_1 (v_2 + \sigma_2 \frac{h_2}{2}))}{\varepsilon}} e^{\frac{ih_2 \cdot (x_2 - x_3)}{\varepsilon}} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\
& J(x_1 + \varepsilon s_1 v_1, v_1 + \sigma_1 \frac{\varepsilon \xi_1 - h_2}{2}) f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_3, v_1, v_2, v_3).
\end{aligned}$$

Write out the phase,

$$\begin{aligned}
&= \frac{1}{\varepsilon^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{\frac{i\varepsilon \xi_1 \cdot (x_1 + \varepsilon s_1 v_1 - x_2 - \varepsilon s_1 (v_2 + \sigma_2 \frac{h_2}{2}))}{\varepsilon}} e^{\frac{-ih_2 \cdot (\varepsilon s_1 v_1 - \varepsilon s_1 (v_2 + \sigma_2 \frac{h_2}{2}))}{\varepsilon}} e^{\frac{-ih_2 \cdot (x_1 - x_2)}{\varepsilon}} e^{\frac{ih_2 \cdot (x_2 - x_3)}{\varepsilon}} \\
& \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) J(x_1 + \varepsilon s_1 v_1, v_1 + \sigma_1 \frac{\varepsilon \xi_1 - h_2}{2}) \\
& f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_3, v_1, v_2, v_3),
\end{aligned}$$

that is,

$$\begin{aligned}
&= \frac{1}{\varepsilon^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\
& e^{i\varepsilon \xi_1 \cdot (x_1 + \varepsilon s_1 v_1 - x_2 - \varepsilon s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} e^{-ih_2 \cdot (s_1 v_1 - s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} e^{\frac{ih_2 \cdot (2x_2 - x_1 - x_3)}{\varepsilon}} \\
& \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) J(x_1 + \varepsilon s_1 v_1, v_1 + \sigma_1 \frac{\varepsilon \xi_1 - h_2}{2}) \\
& f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_3, v_1, v_2, v_3).
\end{aligned}$$

Another change of variable

$$x_{3,old} = 2x_2 - x_1 - \varepsilon x_{3,new},$$

takes us to

$$\begin{aligned} & \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 \\ &= \frac{\varepsilon^3}{\varepsilon^3} \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\ & \quad e^{i\xi_1 \cdot (x_1 + \varepsilon s_1 v_1 - x_2 - \varepsilon s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} e^{-ih_2 \cdot (s_1 v_1 - s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} e^{ih_2 \cdot x_3} \\ & \quad \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) J(x_1 + \varepsilon s_1 v_1, v_1 + \sigma_1 \frac{\varepsilon \xi_1 - h_2}{2}) \\ & \quad f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, 2x_2 - x_1 - \varepsilon x_3, v_1, v_2, v_3). \end{aligned} \quad (2.12)$$

Putting the $\varepsilon \rightarrow 0$ limit inside, which is justified in §2.3, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 \\ &= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^\infty ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\ & \quad e^{i\xi_1 \cdot (x_1 - x_2)} e^{-ih_2 \cdot (s_1 v_1 - s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} e^{ih_2 \cdot x_3} \hat{\phi}(-h_2) \hat{\phi}(h_2) J(x_1, v_1 - \sigma_1 \frac{h_2}{2}) \\ & \quad f^{(3)}(t_2, x_1, x_2, 2x_2 - x_1, v_1, v_2, v_3) \\ &= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_3 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\ & \quad e^{i\xi_1 \cdot (x_1 - x_2)} e^{-ih_2 \cdot (s_1 v_1 - s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} \delta(h_2) \hat{\phi}(-h_2) \hat{\phi}(h_2) \\ & \quad J(x_1, v_1 - \sigma_1 \frac{h_2}{2}) f^{(3)}(t_2, x_1, x_2, 2x_2 - x_1, v_1, v_2, v_3) \end{aligned} \quad (2.13)$$

which is zero under the same reasoning as in the treatment of $Q_{1,3}^{\varepsilon,1} f_N^{(3)}$.

At this point, we have proven that the possible cubic term

$$V = \frac{N^2}{\varepsilon} \int_0^{t_1} S^{(1)}(t_1 - t_2) B_\varepsilon^{(2)} \int_0^{t_2} S^{(2)}(t_2 - t_3) B_\varepsilon^{(3)} f_N^{(3)}(t_3) dt_3 dt_2$$

is zero in the $\varepsilon \rightarrow 0$ limit. Therefore, we have proven that relation (2.1) holds for $f^{(k)}$ and hence established Theorem 1. The rest of this section is to prove that we can take the limits inside the integrals under the assumptions of Theorem 1.

2.3. Justifying $\lim_{\varepsilon \rightarrow 0} \int = \int \lim_{\varepsilon \rightarrow 0}$. We interchanged " $\lim_{\varepsilon \rightarrow 0}$ " and " \int " in going from going from (2.5) to (2.6), from (2.10) to (2.11), and from (2.12) to (2.13). We justify (2.12) to (2.13). The proof of the other two is similar.

We claim that, if

$$\sup_N \sum_{j=1}^3 \sum_{m=0}^4 \left(\left\| \partial_{x_j}^m f_N^{(3)}(t, \cdot) \right\|_{L^1} + \left\| \partial_{v_j}^m f_N^{(3)}(t, \cdot) \right\|_{L^1} \right) < +\infty,$$

then let $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 \\ &= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^\infty ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\ & \quad e^{i\xi_1 \cdot (x_1 - x_2)} e^{-ih_2 \cdot (s_1 v_1 - s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} e^{ih_2 \cdot x_3} \hat{\phi}(-h_2) \hat{\phi}(h_2) J(x_1, v_1 - \sigma_1 \frac{h_2}{2}) \\ & \quad f^{(3)}(t_2, x_1, x_2, 2x_2 - x_1, v_1, v_2, v_3) \end{aligned}$$

if $\int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1$ is given by (2.10).

In fact, rewrite

$$\int J(x_1, v_1) Q_{1,3}^{\varepsilon,2} f_N^{(3)} dx_1 dv_1 = \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 A_\varepsilon(s_1, \xi_1)$$

where

$$\begin{aligned} A_\varepsilon(s_1, \xi_1) &\equiv \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_{\mathbb{R}^3} dh_2 \\ &\quad e^{i\xi_1 \cdot (x_1 + \varepsilon s_1 v_1 - x_2 - \varepsilon s_1 (v_2 + \sigma_2 \frac{h_2}{2}))} e^{-ih_2 \cdot s_1 [v_1 - (v_2 + \sigma_2 \frac{h_2}{2})]} e^{ih_2 \cdot x_3} \\ &\quad \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) J(x_1 + \varepsilon s_1 v_1, v_1 + \sigma_1 \frac{\varepsilon \xi_1 - h_2}{2}) \\ &\quad f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, 2x_2 - x_1 - \varepsilon x_3, v_1, v_2, v_3). \end{aligned}$$

Let $x_1 - x_2 = y_1$, $x_1 + x_2 = y_2$ as well as

$$v_1 - v_2 - \sigma_2 \frac{h_2}{2} = w_1, \quad v_1 + v_2 - \sigma_2 \frac{h_2}{2} = w_2, \quad h_2 = h_2,$$

which makes

$$v_1 = \frac{w_1 + w_2 - 2\sigma_2 h_2}{2}, \quad v_2 = \frac{w_1 - w_2}{2}, \quad h_2 = h_2,$$

we can then transform $A_\varepsilon(s_1, \xi_1)$ into

$$\begin{aligned} &A_\varepsilon(s_1, \xi_1) \\ &= \int e^{i\xi_1 \cdot \{y_1 + \varepsilon s_1 [\frac{w_1 + w_2 - 2\sigma_2 h_2}{2}] - \varepsilon s_1 (\frac{w_1 - w_2}{2} + \sigma_2 \frac{h_2}{2})\}} e^{-ih_2 \cdot s_1 w_1} e^{ih_2 \cdot x_3} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\ &\quad J(\frac{y_1 + y_2}{2} + \varepsilon s_1 \frac{w_1 + w_2 - 2\sigma_2 h_2}{2}, \frac{w_1 + w_2 - 2\sigma_2 h_2}{2} + \sigma_1 \frac{\varepsilon \xi_1 - h_2}{2}) \\ &\quad f_N^{(3)}(t_2 - \varepsilon s_1, \frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}, \frac{y_1}{2} - \frac{3y_2}{2} - \varepsilon x_3, \frac{w_1 + w_2 - 2\sigma_2 h_2}{2}, \frac{w_1 - w_2}{2}, v_3). \\ &\equiv \int e^{i\xi_1 \cdot y_1} e^{-ih_2 \cdot s_1 w_1} e^{ih_2 \cdot x_3} B, \end{aligned}$$

where $B(\varepsilon s_1, y_1, y_2, \varepsilon x_3, w_1, w_2, \varepsilon \xi_1, h_2, v_3) \equiv$

$$\begin{aligned} &e^{i\xi_1 \cdot \varepsilon s_1 \{[\frac{w_1 + w_2 - 2\sigma_2 h_2}{2}] - (\frac{w_1 - w_2}{2} + \sigma_2 \frac{h_2}{2})\}} \hat{\phi}(\varepsilon \xi_1 - h_2) \hat{\phi}(h_2) \\ &J(\frac{y_1 + y_2}{2} + \varepsilon s_1 \frac{w_1 + w_2 - 2\sigma_2 h_2}{2}, \frac{w_1 + w_2 - 2\sigma_2 h_2}{2} + \sigma_1 \frac{\varepsilon \xi_1 - h_2}{2}) \\ &f_N^{(3)}(t_2 - \varepsilon s_1, \frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}, \frac{y_1}{2} - \frac{3y_2}{2} - \varepsilon x_3, \frac{w_1 + w_2 - 2\sigma_2 h_2}{2}, \frac{w_1 - w_2}{2}, v_3) \end{aligned} \quad (2.14)$$

Clearly, for bounded ξ_1, x_3 and s_1 , such a integral is finite. We only need to control large ξ_1, x_3 and s_1 to pass to the limit.

Upon using standard smooth cutoff functions, we only need to concentrate the most singular region of

$$|\xi_1| \geq 1, |x_3| \geq 1, |s_1| \geq 1. \quad (2.15)$$

we may further assume that in such a region,

$$|\xi_1^1| \gtrsim |\xi_1|, \quad |x_3^1| \gtrsim |x_3|, \quad |h_2^1| \gtrsim |h_2| \quad (2.16)$$

All the other cases are simpler and can be controlled similarly.

We first integrate by part in y_1^1, w_1^1 repeatedly, (since εs_1 is bounded), to obtain

$$\begin{aligned} &\int dy_1 dy_2 dx_3 dw_1 dw_2 dv_3 dh_2 \frac{1}{\{\xi_1^1\}^m s_1^m \{h_2^1\}^m} e^{i\xi_1 \cdot y_1} e^{-ih_2 \cdot s_1 w_1} e^{ih_2 \cdot x_3} \\ &\partial_{y_1^1}^m \partial_{w_1^1}^m B(\varepsilon s_1, y_1, y_2, x_3, w_1, w_2, \xi_1, h_2, v_3) \\ &= \int dy_1 dy_2 dx_3 dw_1 dw_2 dv_3 dh_2 \frac{1}{\{\xi_1^1\}^m s_1^m \{h_2^1\}^m} e^{i\xi_1 \cdot y_1} e^{-ih_2 \cdot s_1 w_1} \\ &\quad \frac{1}{x_3^1} \frac{d\{e^{ih_2^1 x_3^1}\}}{dh_2^1} \partial_{y_1^1}^m \partial_{w_1^1}^m B(\varepsilon s_1, y_1, y_2, x_3, w_1, w_2, \xi_1, h_2, v_3). \end{aligned}$$

we then take integration by part in h_2^1 four times as above to get the worse term, in terms of vanishing order of $\hat{\phi}(\varepsilon\xi_1 - h_2)\hat{\phi}(h_2)$ in B as

$$\begin{aligned} & \sim \sum_j \int dy_1 dy_2 dx_3 dw_1 dw_2 dv_3 dh_2 \frac{s_1^j \{w_1^1\}^j}{\{\xi_1^1\}^m s_1^m \{h_2^1\}^m \{x_1^1\}^j} e^{i\xi_1 \cdot y_1} e^{-ih_2 \cdot s_1 w_1} e^{ih_2 \cdot x_3} \\ & \quad \partial_{h_2^1}^{4-j} \partial_{h_2^1}^m \partial_{y_1^1}^m \partial_{w_1^1}^m B(\varepsilon s_1, y_1, y_2, \varepsilon x_3, w_1, w_2, \varepsilon \xi_1, h_2, v_3) \\ & \sim \sum_{j=0}^m \int dy_1 dy_2 dx_3 dw_1 dw_2 dv_3 dh_2 \frac{s_1^j \{w_1^1\}^j}{\{\xi_1^1\}^m s_1^m \{h_2^1\}^m \{x_1^1\}^j} e^{i\xi_1 \cdot y_1} e^{-ih_2 \cdot s_1 w_1} e^{ih_2 \cdot x_3} \\ & \quad \partial_{h_2^1}^{4-j+m} [\hat{\phi}(h_2) \hat{\phi}(\varepsilon\xi_1 - h_2)] \times B_j(\varepsilon s_1, y_1, y_2, \varepsilon x_3, w_1, w_2, \varepsilon \xi_1, h_2, v_3). \end{aligned}$$

Here B_j is some nice function with decay in w_2 so that the growth in w_1^1 is under control. Hence, this is uniformly integrable for large ξ_1, s_1, x_3, h_2 if $m > 5$ by (2.16) and (2.15). It suffices to control small h_2^1 for $|h_2^1| < 1$.

We now use the vanishing condition of $\hat{\phi}$: $\hat{\phi}(h_2) = h_2^n$ for $|h_2| \leq 1$, so that for $0 \leq j \leq 4$,

$$|\partial_{h_2^1}^{m+4-j} [\hat{\phi}(h_2) \hat{\phi}(\varepsilon\xi_1 - h_2)]| \leq h_2^{n-m-4}.$$

Hence by (2.16) near $h_2 = 0$, the integral has a singularity of $\frac{1}{h_2^{4-n+2m}}$. If $4 - n + 2m < 3$, or $n \geq 1 + 2m > 11$, then we know that $\int_{|h_2| \leq 1} \frac{1}{h_2^{4-n+2m}} dh_2 < +\infty$, and it is uniformly bounded integrable, by (2.16). Hence, we can interchange "lim $_{\varepsilon \rightarrow 0}$ " and " \int " in going from (2.12) to (2.13) as claimed.

2.4. It is Necessary to Have $\int \phi = 0$. Recall (2.10),

$$\begin{aligned} & \int J(x_1, v_1) Q_{1,3}^{\varepsilon,1} f_N^{(3)} dx_1 dv_1 \\ &= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_3 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\ & \quad e^{i\xi_1 \cdot (x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - \varepsilon s_1 v_2)} e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} e^{ih_2 \cdot x_3} \hat{\phi}(\varepsilon\xi_1 - h_2) \hat{\phi}(h_2) \\ & \quad J(x_1 + \varepsilon s_1 \left(v_1 + \sigma_2 \frac{h_2}{2} \right), v_1 + \sigma_2 \frac{h_2}{2} - \sigma_1 \frac{h_2}{2} + \sigma_1 \frac{\varepsilon \xi_1}{2}) \\ & \quad f_N^{(3)}(t_2 - \varepsilon s_1, x_1, x_2, x_2 - \varepsilon x_3, v_1, v_2, v_3). \end{aligned} \tag{2.17}$$

To see that it is necessary to have $\int \phi = 0$ in order to have a $\varepsilon \rightarrow 0$ limit for the BBGKY hierarchy (1.1) and hence a possible derivation for the quantum Boltzmann hierarchy (1.4) and the quantum Boltzmann equation (1.6), we analysis the size of (2.17). To avoid some technical issues, let us assume that $J(\dots) f_N^{(3)}(\dots)$ is a test function g , because the main point here is the phase

$$e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} e^{ih_2 \cdot x_3}.$$

Do the dx_3 integrals in (2.17), we have

$$\begin{aligned} &= \sum_{\sigma_1, \sigma_2 = \pm 1} \sigma_1 \sigma_2 \int d\mathbf{x}_2 \int d\mathbf{v}_3 \int_0^{\frac{t_2}{\varepsilon}} ds_1 \int_{\mathbb{R}^3} d\xi_1 \int_{\mathbb{R}^3} dh_2 \\ & \quad e^{i\xi_1 \cdot (x_1 + \varepsilon s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - x_2 - \varepsilon s_1 v_2)} e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} \\ & \quad \hat{\phi}(\varepsilon\xi_1 - h_2) \hat{\phi}(h_2) \frac{1}{\varepsilon^3} \hat{g}(t_2 - \varepsilon s_1, x_1, x_2, \frac{h_2}{\varepsilon}, v_1, v_2, v_3). \end{aligned}$$

Here \hat{g} means the Fourier transform in x_3 . We then find that, for every $t_2, \mathbf{x}_2, \mathbf{v}_3$, we effectively have a $\delta(h_2)$, so that the dh_2 integral is restricted to have size $|h_2| \lesssim \varepsilon$. Now, say $t_2 \lesssim 1$, we know $|s_1 h_2| \lesssim 1$ and hence

$$e^{-ih_2 \cdot (s_1 (v_1 + \sigma_2 \frac{h_2}{2}) - s_1 v_2)} \sim 1$$

which then makes the ds_1 integral to blow up as $\varepsilon \rightarrow 0$ if $\hat{\phi}(0) \neq 0$.

APPENDIX A. THE CUBIC TERM OF (1.3) WHEN $\hat{\phi}(0) = 0$

Theorem 1 is unexpected. It rules out the possibility to have the Uehling-Uhlenbeck equation (1.3) as the mean-field equation from the BBGKY hierarchy (1.1) in the sense that if f solves (1.3), then

$$f^{(k)}(t, \mathbf{x}_k, \mathbf{v}_k) = \prod_{j=1}^k f(t, x_j, v_j)$$

is not a solution to the $N \rightarrow \infty$ limit of the BBGKY hierarchy (1.1). It is then natural to wonder if the assumption: $\hat{\phi}(0) = 0$, implies that the cubic term in the Uehling-Uhlenbeck equation (1.3) is zero. Such an statement is unlikely to be true. We include a discussion here for completeness. On the one hand, if $\hat{\phi}(0) \neq 0$, the $\varepsilon \rightarrow 0$ limit for the BBGKY hierarchy (1.1) has an infinite cubic term as shown formally in the proof of Theorem 1 and in §2.4. On the other hand, recall the cubic term in (1.3)

$$M(f) = 8\pi^3\theta \int dv_* \int dv'_* \int dv' W(v, v'|v_*, v'_*) [(f' f'_* f + f' f'_* f_*) - (f f_* f'_* + f f_* f')],$$

and

$$\begin{aligned} W(v, v'|v_*, v'_*) &= \frac{1}{8\pi^2} \left[\hat{\phi}(v' - v) + \theta \hat{\phi}(v' - v_*) \right]^2 \delta(v + v_* - v' - v'_*) \\ &\quad \times \delta\left(\frac{1}{2} (v^2 + v_*^2 - (v')^2 - (v'_*)^2)\right). \end{aligned}$$

Let us suppress the (t, x) dependence in f, f', f_*, f'_* and write

$$f = f(v), \quad f' = f(v'), \quad f_* = f(v_*), \quad f'_* = f(v'_*).$$

since the integral we are considering has nothing to do with that. With the usual parametrization:

$$v' = v + [(v - v_*) \cdot \omega] \omega, \quad v'_* = v_* - [(v - v_*) \cdot \omega] \omega,$$

we reach

$$\begin{aligned} M(f) &= \pi\theta \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} dS_\omega |v - v_*| \\ &\quad \times \left[\hat{\phi}([(v - v_*) \cdot \omega] \omega) + \theta \hat{\phi}((v - v_*) + [(v - v_*) \cdot \omega] \omega) \right]^2 \\ &\quad \times [f' f'_* (f + f_*) - f f_* (f' + f'_*)]. \end{aligned}$$

Let $\theta = 1$. Assume that $\hat{\phi}$ does not change sign and $\hat{\phi}(\xi) = 0$ only at $\xi = 0^2$, then

$$|v - v_*| \left[\hat{\phi}([(v - v_*) \cdot \omega] \omega) + \theta \hat{\phi}((v - v_*) + [(v - v_*) \cdot \omega] \omega) \right]^2 = 0$$

only when $v_* = v$ which is a measure zero set. It is then hard to believe that if $\hat{\phi}(\xi) = 0$ only at $\xi = 0$ will make $M(f) = 0$ for every f .

REFERENCES

- [1] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, *Some considerations on the derivation of the nonlinear quantum boltzmann equation*, J. Stat. Phys. **116** (2004), 381–410.
- [2] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, *On the weak-coupling limit for bosons and fermions*, Math. Mod. Meth. Appl. Sci. **15** (2005), 1811–1843.
- [3] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, *From the N-body Schroedinger equation to the quantum Boltzmann equation: a term-by-term convergence result in the weak coupling regime*, Commun. Math. Phys. **277** (2008), 1–44.
- [4] L. Erdős, M. Salmhofer and H. T. Yau, *On the Quantum Boltzmann Equation*, J. Stat. Phys. **116** (2004) 367–380.
- [5] I. Gallagher, L. Saint-Raymond, and B. Texier, *From Newton to Boltzmann: hard spheres and short-range potentials*, arXiv:1208.5753.
- [6] F. King, *BBGKY Hierarchy for Positive Potentials*, PhD Thesis, Dept. Mathematics, Univ. California, Berkley, 1975.
- [7] O. E. Lanford III, *Time Evolution of Large Classical Systems*, Lecture Notes in Physics, **38** (1975), 1–111.
- [8] E. A. Uehling and G. E. Uhlenbeck, *Transport Phenomena in Einstein–Bose and Fermi–Dirac Gases*, Phys. Rev. **43** (1933) 552–561.

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, 151 THAYER STREET, PROVIDENCE, RI 02912

E-mail address: chenxuwen@math.brown.edu

URL: <http://www.math.brown.edu/~chenxuwen/>

DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, 182 GEORGE STREET, PROVIDENCE, RI 02912

E-mail address: yan.guo@brown.edu

URL: <http://www.cfm.brown.edu/people/guoy/>

²Such a potential ϕ could be constructed.